

# Zero-Temperature Limit of the SUSY-breaking Complexity in Diluted Spin-Glass Models

Giorgio Parisi<sup>1</sup> and Tommaso Rizzo<sup>2</sup>

<sup>1</sup>*Dipartimento di Fisica, Università di Roma "La Sapienza",  
INFN and Center for Statistical Mechanics and Complexity,  
INFM Roma 1 P.le Aldo Moro 2, 00185 Roma, Italy*

<sup>2</sup>*Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris, France*

We study the SUSY-breaking complexity of the Bethe Lattice Spin-Glass in the zero temperature limit. We consider both the Gaussian and the bimodal distribution of the coupling constants. For  $J_{ij} = \pm 1$  the SUSY breaking theory yields fields distributions that concentrate on integer values at low temperatures, at variance with the unbroken SUSY theory. This concentration takes place both in the quenched as well as in the simpler annealed formulation.

The complexity, *i.e.* the logarithm of the number of metastable states, is a crucial tool to understand spin-glasses and related disordered systems [1]. Recently it became clear that in mean-field models there are two possibilities: a supersymmetric (SUSY) theory and a spontaneously SUSY-broken theory. The behavior of  $\Sigma(f)$ , the complexity of the states with free energy  $f$ , in one-step replica-symmetry-breaking (1RSB) models is the following: at low free energies (including the equilibrium one) the theory is supersymmetric (SUSY) while at higher free energies the SUSY-breaking theory must be used [2]. In models with full-RSB (FRSB) like the Sherrington-Kirkpatrick model the SUSY solution is unstable immediately above the equilibrium free energy  $f_{eq}$  [3] and the complexity must be computed using a SUSY-breaking approach at all  $f > f_{eq}$ . The paradigm of the SUSY-breaking solution is the Bray and Moore computation of the SK model that in the replica formulation corresponds to the two-group ansatz [4]. Some apparent inconsistencies of this theory [5] have been solved by the discovery of a vanishing isolated eigenvalue [6] in the spectrum of the TAP Hessian and in [7] it has been proved that this result is a consequence of SUSY breaking and therefore is very general. Later these results have also received some numerical confirmations [8]. Thus the relevant states are marginal within the SUSY-breaking theory but there are also non-marginal states with a lower total complexity [9]. The theory of diluted spin-glass models and optimization problems has received a considerable boost by the application of the cavity method [10, 12]. The standard computation scheme of the complexity used in this context [13] corresponds to the SUSY theory and becomes unstable in certain regions of the parameter space [14]. Much as in mean-field models with infinite connectivity, this instability could be caused either by a 1RSB/FRSB transition or by a less standard SUSY/SUSY-breaking transition. This has motivated the search for a formulation of the SUSY-breaking theory within the cavity method which has been achieved in [9], see also [15]. If we add a small perturbation to the weight of the states [9], we select a set of non-marginal states slightly different from the relevant ones, and the cavity method can be applied. The relevant states are

recovered sending to zero the perturbation; this limit is non-trivial, indeed as the perturbation goes to zero the reweighing associated to its presence remains finite because of marginality. Later this method has been used to derive the SUSY breaking theory for diluted models [16]. Here we study the zero temperature limit of this theory which is particularly interesting for the application to optimization models. Particularly in the case of bimodal distribution of the coupling constants this limit is rather non-trivial, and our main result is that the field distributions concentrate over integer values. Remarkably this happens already within the so-called factorized approximation, a result not totally unexpected because it has been recognized that it corresponds to the annealed solution.

*The Complexity at Finite Temperature* - In the following we recall the theory at finite temperature [16]. One finds the recursive equations for the fields  $h$  and  $z$  in the Bethe lattice spin-glass are:

$$h_0 = \sum_{i=1}^k u(J_i, h_i) \quad ; \quad z_0 = \sum_{i=1}^k \frac{du(J_i, h_i)}{dh_i} z_i \quad (1)$$

where  $u(J, h) \equiv \beta^{-1} \operatorname{arctanh}[\tanh(\beta J) \tanh(\beta h)]$ .

The complexity is the number of the solutions of these equations. The fields  $z_i$  are zero in the standard SUSY solution, they are non-zero only if the solutions are marginally stable (it can be argued that they are proportional to the components of the eigenvector of the vanishing isolated eigenvalue).

The recursive equation for the distribution of the fields  $h_0$  and  $z_0$  is

$$P_0(h_0, z_0) = C \int \prod_{i=1}^k P_i(h_i, z_i) e^{u\beta\Delta F^{(iter)} + \Delta X^{(iter)}} \times \\ \times \delta\left(h_0 - \sum_{i=1}^k u(J_i, h_i)\right) \delta\left(z_0 - \sum_{i=1}^k \frac{du(J_i, h_i)}{dh_i} z_i\right) dh_i dz_i, \quad (2)$$

where:

$$-\beta\Delta F^{(iter)} = \ln[2 \cosh \beta h_0] + \sum_{i=1}^k \ln \left[ \frac{\cosh \beta J_i}{\cosh \beta u(J_i, h_i)} \right] \quad (3)$$

$$\Delta X^{(iter)} = \sum_{i=1}^k [\tanh \beta h_0 - \tanh \beta J_i \tanh \beta h_i] \frac{du}{dh_i} z_i \quad (4)$$

The shifts in the process of merging  $k+1$  lines,  $\Delta F^{(1)}$  and  $\Delta X^{(1)}$ , are the same as above with  $k$  replaced by  $k+1$ . The distribution of distributions  $\mathcal{P}[P(h, z)]$  can be determined numerically by population dynamics [10]; as usual in the factorized approximation the population  $P_i(h_i, z_i)$  is site independent and corresponds to the annealed theory [16]. The functional  $\Phi(u) = \Sigma(f) - u\beta f$  is the Legendre transform of the complexity; its shift is given by:

$$\Delta\Phi(u)^{(iter)} = \ln \int \prod_{i=1}^k P_i(h_i, z_i) dh_i dz_i e^{u\beta\Delta F^{(iter)} + \Delta X^{(iter)}} \quad (5)$$

The Legendre transform with respect to  $u$  of the complexity is given by

$$\begin{aligned} \Phi(u) &= \frac{k+1}{2} \int \prod_{i=1}^k \mathcal{P}(P_i(h_i, z_i)) \langle \Delta\Phi(u)^{(iter)} \rangle_J \\ &\quad - \frac{k-1}{2} \int \prod_{i=1}^{k+1} \mathcal{P}(P_i(h_i, z_i)) \langle \Delta\Phi(u)^{(1)} \rangle_J \end{aligned} \quad (6)$$

where the square brackets mean average over the bonds  $J_i$ , and  $\Delta\Phi^{(1)}$  has the same expression  $\Delta\Phi^{(iter)}$  with  $k+1$  indexes instead of  $k$ . At  $u=0$  eq. (6) gives the total complexity. The free energy is given by  $\beta f = d\Phi/du$ . This derivative can be computed numerically or more efficiently by deriving with respect to  $u$  the variational expression of  $\Phi(u)$ , eq. 35 in [16].

*The Zero-Temperature Limit - Gaussian Distribution of the  $J_{ij}$*  - In the  $T \rightarrow 0$  limit we have to modify the functions  $u(h_i, J_i)$  and  $du/dh$  entering the recursion equations for the fields,

$$\lim_{T \rightarrow 0} u(J, h) = \text{sign } J_i \begin{cases} h_i & \text{if } |h_i| < |J_i| \\ |J_i| \text{sign } h_i & \text{if } |h_i| > |J_i| \end{cases} \quad (7)$$

$$\lim_{T \rightarrow 0} \frac{du}{dh} = \text{sign } J_i \begin{cases} 1 & \text{if } |h_i| < |J_i| \\ 0 & \text{if } |h_i| > |J_i| \end{cases} \quad (8)$$

The limit of the shifts are obtained simply replacing the functions  $\tanh \beta h_i$  with  $\text{sign } h_i$  in the expressions (3) and (4), thus we obtain

$$-\Delta F^{(1)} = |h_0| + \sum_{i=1}^{k+1} (|J_i| - |u(J_i, h_i)|) \quad (9)$$

$$\Delta X^{(1)} = \sum_{i=1}^{k+1} [\text{sign } h_0 - \text{sign}(J_i h_i)] \frac{du}{dh_i} z_i \quad (10)$$

Using this expression we can evaluate the distributions of the fields at zero temperature through population dynamics. In general the resulting distributions of the fields have the following properties: i) there is a finite probability of having a strictly zero value of  $z$ , ii) the non-zero values of  $z$  are rather large and diverging with the connectivity, iii) there is a strong correlation between  $z_i$  and  $h_i$  such that  $\text{sign}(h_i z_i)$  has a very small probability of being negative.

*Bimodal Distribution ( $J_{ij} = \pm 1$ )*. In this case the fields concentrate over the integers in the  $T \rightarrow 0$  limit but we must keep track of the behavior of the finite-temperature corrections to the integer value. We rewrite the field  $h_i$  as the sum of an integer contribution  $k_i$  and an infinitesimal contribution  $T\epsilon_i$ :  $h_i = k_i + T\epsilon_i + O(T^2)$ . The fields  $\epsilon_i$  remain finite in the  $T \rightarrow 0$  limit. We must consider populations of triplets  $\{k_i, \epsilon_i, z_i\}$  obeying the following recursion equations:

$$k_0 = \sum_{i=1}^k J_i \text{sign}(k_i) \quad (11)$$

$$\epsilon_0 = \sum_{i=1}^k \text{sign}(J_i) g(k_i, \epsilon_i) \quad (12)$$

$$z_0 = \sum_i^k \frac{du}{dh_i} z_i = \sum_i^k \left| \frac{du}{dh_i} \right| \text{sign}(J_i) z_i \quad (13)$$

where the function  $g(k_i, \epsilon_i)$  can be obtained taking the limit of the expression of  $u(J_i, h_i)$  and is given by:

$$g(k_i, \epsilon_i) = \begin{cases} \epsilon_i & \text{for } k_i = 0 \\ \frac{\epsilon_i}{2} - \frac{\text{sign}(k_i)}{2} \ln 2 \cosh \epsilon_i & \text{for } |k_i| = 1 \\ 0 & \text{for } |k_i| > 1 \end{cases} \quad (14)$$

$$\left| \frac{du}{dh_i} \right| = \begin{cases} 1 & \text{for } k_i = 0 \\ \frac{1}{2} - \frac{\text{sign}(k_i)}{2} \tanh \epsilon_i & \text{for } |k_i| = 1 \\ e^{-2\beta(|h_i| - |J_i|)} \simeq 0 & \text{for } |k_i| > 1 \end{cases} \quad (15)$$

The  $\epsilon_i$ 's have a precise meaning at zero temperature: if the spin 0 is a *spin-four*, i.e.  $k_0 = 0$  the quantity  $\tanh \epsilon_0$  is its magnetization averaged over the many configurations of the same energy that represent the zero temperature state. At low temperatures the behavior of the  $z$  fields is dramatically different depending on the parity of  $k$ . If  $k$  is even, the  $z$  fields remains finite in the  $T \rightarrow 0$  limit (they would diverge in the high-connectivity limit). Instead if  $k$  is odd, the  $z$ -fields diverge as  $\exp(2\beta|J|)$  and must be rescaled at zero temperature.

*Even Values of  $k$*  - In this case the  $z$ -fields remain finite. In order to compute the reweighing term eq. (4) in the zero temperature limit for  $k$  even it is sufficient to take the limit of  $\tanh(\beta h_i)$  which reads:

$$\lim_{T \rightarrow 0} \tanh \beta h_i = \begin{cases} \text{sign } k_i & \text{for } k_i \neq 0 \\ \tanh \epsilon_i & \text{for } k_i = 0 \end{cases} \quad (16)$$

The free energy shift reads [11]

$$-\Delta F^{(1)} = |k_0| + \sum_{i=1}^{k+1} (|J_i| - |\text{sign} k_i|). \quad (17)$$

We have determined the distribution of the fields by evolving numerically populations of triplets  $\{k_i, e_i, z_i\}$  at  $u = 0$ . For generic even values of  $k$  the resulting distribution of the fields is such that: i) the  $z$ 's have a finite probability of being exactly zero, ii) The non-zero values of  $z$  are rather large and diverging with the connectivity  $k$ , iii) there is a strong correlation between  $z_i$  and  $h_i$  such that  $\text{sign}(h_i z_i)$  has a very small probability of being negative, iiiii) the entropic corrections  $\epsilon_i$  are rather large, (see fig. 1).

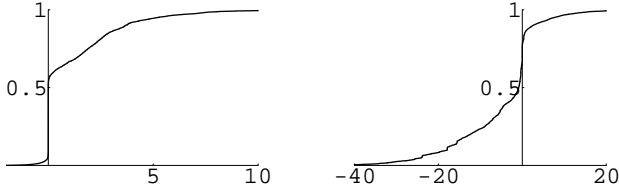


FIG. 1:  $T \rightarrow 0$  limit annealed theory for Bimodal Distribution,  $k=2$ . Left: cumulative distribution function of  $\text{sign}(k_i)z_i$ , Right: cumulative distribution function of the  $\epsilon_i$  at  $k_i = 1$

*Odd Values of  $k$*  - In this case the fields generated in the merging of  $k$  branches are concentrated over odd values of the integers. In particular they cannot be equal to zero and the term  $\tanh \beta h_i$  will be always equal to  $\pm 1$ . This has deep consequences on the reweighing term, eq. (4), and leads to the divergence of the  $z$ 'fields. For  $k$ 's different from zero eq. (4) can be rewritten as the sum over  $i$  terms of the form:

$$\Delta X_i = \left( \text{sign}(h_0)(1 - 2e^{-2\beta|h_0|}) - \text{sign}(J_i)\text{sign}(h_i) \times \right. \\ \left. \times (1 - 2e^{-2\beta|J_i|} - 2e^{-2\beta|h_i|}) \right) \left| \frac{du}{dh_i} \right| \text{sign}(J_i)z_i \quad (18)$$

This term is zero if  $z_i = 0$ , while if  $z_i \neq 0$  it can be finite, null or equal to  $-\infty$ , thus leading to a vanishing weight. In order to analyze these possibilities it is useful to distinguish two cases. In the first case we have:

$$\text{sign}(h_0)\text{sign}(J_i)\text{sign}(h_i) = -1 \quad (19)$$

Then the first factor in (18) is finite and equal to  $-2\text{sign}(J_i)\text{sign}(h_i)$  therefore

$$\Delta X_i = -2\text{sign}(h_i) \left| \frac{du}{dh_i} \right| z_i \quad (20)$$

Now we make the crucial hypotheses that  $z_i$  either is zero (with finite probability) or infinite (more precisely

$O(e^{2\beta|J_i|})$ ) and in this case it has the same sign as  $h_i$ . These hypotheses are motivated by the behavior of the various quantities at finite low temperature observed numerically. At the end we will check that these hypotheses are self-consistent. From eq. (15) we find that:

$$\Delta X_i = \begin{cases} O(e^{-2\beta(|h_i|-|J_i|-1)}) \simeq -\infty & \text{for } |k_i| = 1, z \neq 0 \\ O(e^{-2\beta(|h_i|-|J_i|-1)}) \simeq 0 & \text{for } |k_i| > 1, z \neq 0 \end{cases}$$

The last equation follows from the fact that the smaller value of  $|k_i| > 1$  is  $|k_i| = 3$  because of the odd value of  $k$ . In the second case we have:

$$\text{sign}(h_0)\text{sign}(J_i)\text{sign}(h_i) = 1$$

Then the first factor in (18) is infinitesimal and  $\Delta X_i$  is given by:

$$\Delta X_i = 2 \left( e^{-2\beta|J_i|} - e^{-2\beta|h_0|} + e^{-2\beta|h_i|} \right) \left| \frac{du}{dh_i} \right| \text{sign}(h_i)z_i \quad (21)$$

We note that the first term is order  $e^{-2\beta|J_i|}$  therefore

$$\Delta X_i = O(e^{-2\beta(|h_i|-1)}) \simeq 0 \quad \text{for } z \neq 0, |k_i| > 1$$

Finally we consider the case  $|k_i| = 1$ . This is the only case in which we have a finite reweighing. To rewrite eq. (21) we note that the sign of  $h_i$  is simply given by the sign of  $k_i$  thus we have:

$$|h_i| = |k_i| + T\text{sign}(k_i)\epsilon_i \implies e^{-2\beta|h_i|} = e^{-2\beta|k_i|} e^{-2\text{sign}(k_i)\epsilon_i}$$

The same result applies also to  $h_0$  thus we have:

$$\Delta X_i = 2 \left| \frac{du}{dh_i} \right| \text{sign}(h_i)z_i e^{-2\beta|J_i|} \times \\ \times \begin{cases} 1 + e^{-2\text{sign}(k_i)\epsilon_i} & \text{for } |k_i| = 1, |k_0| > 1 \\ 1 - e^{-2\text{sign}(k_0)\epsilon_0} + e^{-2\text{sign}(k_i)\epsilon_i} & \text{for } |k_i| = 1, |k_0| = 1 \end{cases}$$

The presence of the term  $O(e^{-2\beta|J_i|})$  fixes the scales of the non-zero values of  $z$  to  $O(e^{2\beta|J_i|})$  in agreement with our hypotheses. Therefore we must rescale  $z$  to work directly in  $T \rightarrow 0$  limit. The condition  $\text{sign}(h_i z_i) \geq 0$  is self-consistent under iteration, indeed it can be checked that the reweighing  $\Delta X_i$  gives zero weight to the triplet  $\{k_0, \epsilon_0, z_0\}$  unless each contribution  $du_i/dh_i z_i$  to  $z_0$  has the same sign as  $h_0$ . In order to compute the complexity we must consider also the merging of  $k+1$  branches. The main difference is that in this case  $k_0$  can be equal to zero. First of all we note that if  $k_0 \neq 0$  the computation of the anomalous shift  $\Delta X$  is the same as above. The quantity  $\Delta X_i$  in the case that  $k_0 = 0$  can be rewritten as the sum over  $i$  of the following terms in the low temperature limit:

$$\Delta X_i = (\tanh \epsilon_0 - \text{sign}(J_i)\text{sign}(h_i)) \left| \frac{du}{dh_i} \right| \text{sign}(J_i)z_i = \\ = - \left| \frac{du}{dh_i} \right| \text{sign}(h_i)z_i (1 - \text{sign}(J_i h_i) \tanh \epsilon_0) \quad (22)$$

since  $|\tanh \epsilon_0| \leq 1$  the last factor is always positive and

$$\Delta X_i = -\infty \quad \text{for } z_i \neq 0, k_0 = 0 \quad (23)$$

The same result applies when considering the quantities which enter the variational expression of the complexity [16] and of the average free energy. We have determined the distribution of the fields at  $u = 0$  for odd  $k$  by evolving numerically populations of triplets  $\{k_i, e_i, z_i\}$ , where  $z_i$  has been rescaled as  $z_i e^{-2\beta|J|} \rightarrow z_i$  in order to be finite, see fig. (2). As expected there is a finite probability of having  $z_i = 0$ . We stress that the complicated behavior of the odd- $k$  model is determined by the fact that  $k_i$  is always different from zero, therefore more generic situation (*e.g.* fluctuating connectivity) should display a simpler behavior, like the one of the even- $k$  model.

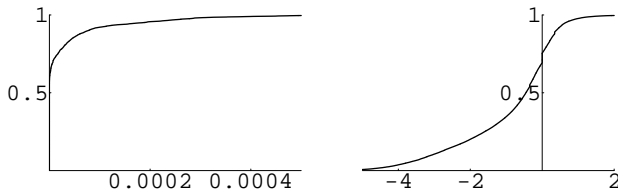


FIG. 2:  $T \rightarrow 0$  limit annealed theory for Bimodal Distribution,  $k=3$ . Left: cumulative distribution function of  $\text{sign}(k_i)z_i$  with rescaled  $z_i$ ; Right: cumulative distribution function of the  $\epsilon_i$  at  $k_i = 1$

In [16] it was found that in the Bethe lattice spin-glass the total complexity (which is obtained setting  $u = 0$ ) is given by the SUSY-breaking theory *i.e.*  $z \neq 0$ . We have checked that this property remains true in the  $T \rightarrow 0$  limit in the three models considered. We also stress that if we force  $z_i = 0$  the population dynamics algorithms of the bimodal case do not converge and in particular the  $\epsilon_i$ 's diverge implying that the decomposition  $h_i = k_i + T\epsilon_i$  breaks down leading to a distribution which is no more concentrated over the integers. The numerical computation of the complexity requires higher computational efforts and is currently underway.

Our results raise an interesting question. In the bimodal case the “strong” variables  $k_i$  obey the recursive

equations eq. (11) that do not depend on the  $e_i$ 's and the  $z_i$ 's. We call the solutions  $\{k_i^\alpha\}$  of these equations “strong belief-propagation solutions” (SBPS). Instead the complexity we are computing yields the number of solutions  $\{k_i^\alpha, e_i^\alpha, z_i^\alpha\}$  of equations (11,12,13), we call them full belief-propagation solutions (FBPS). In the standard theory with  $z_i = 0$  the  $X$  shift is zero and the complexity depends only on the strong variables, thus we obtain for the FBPS complexity the same result we would obtain for the complexity of the SBPS. Instead the fact that when  $z_i \neq 0$  the complexity depends not only on the  $k_i$  but also on the  $\epsilon_i$  and  $z_i$  leads to the possibility that there is no biunivocal correspondence between FBPS and SBPS. This could happen either because for each SBPS there are many FBPS or because there is no FBPS corresponding to a given SBPS.

However it is interesting to notice that the question can be formulated as a percolation problem. Starting from a given SBPS,  $\{k_i^\alpha\}$ , we want to determine a FBPS  $\{\epsilon_i^\alpha, z_i^\alpha\}$  which solves equations (11,12,13). We can proceed in the following way, we consider a root site 0 and we start exploring all the trees starting from it. On each spin  $i$  we measure the field  $k_i$  that it passes in the direction of the root. If  $|k_i| \leq 1$  we go to the next level of the tree while if  $|k_i| > 1$  we stop because from eqs. (14) and (15) we know that it will pass zero messages  $g(k_i, \epsilon_i) = 0$  and  $du_i/dh_i z_i = 0$ . If at a certain point we stop completely, we are left with a tree with vanishing messages passed by the leaves, thus we can compute the unique value  $(\epsilon_0, z_0 = 0)$  in a sweep. If we can repeat the process for each point of the lattice we find that there is a unique FBPS corresponding to the given SBPS and that it has  $z_i = 0 \forall i$ . Thus we can find non vanishing  $z$ -fields only if we have percolation in this process, correspondingly we cannot determine uniquely  $\epsilon_0$  and the biunivocal correspondence between SBPS and FBPS becomes uncertain.

We thank A. Montanari for interesting discussions. Work supported in part by the European Community's Human Potential Program under the project Dyglagemem and Evergrow.

- 
- [1] M. Mézard, G. Parisi, and M.A. Virasoro, “Spin glass theory and beyond”, World Scientific (Singapore 1987).
  - [2] A. Crisanti, L. Leuzzi, and T. Rizzo, cond-mat/ 0406649, A. Annibale, G. Gualdi, A. Cavagna cond-mat/ 0406466.
  - [3] A. Crisanti, L. Leuzzi, G. Parisi, T. Rizzo, Phys. Rev. Lett. **92**, 127203 (2004)
  - [4] A.J. Bray and M.A. Moore, J. Phys. C **13**, L469 (1980).
  - [5] J. Kurchan, J. Phys. A **24** (1991) 4969, A. Cavagna, I. Giardina, G. Parisi and M. Mézard, J. Phys. A **36** (2003) 1175, A. Crisanti, L. Leuzzi, G. Parisi, T. Rizzo, Phys. Rev. B, **68** (2003) 174401
  - [6] T. Aspelmeier, A. J. Bray, and M. A. Moore Phys. Rev. Lett. **92**, 087203 (2004).
  - [7] G. Parisi and T. Rizzo, J. Phys. A **37** (2004) 7979
  - [8] A. Cavagna, I. Giardina, G. Parisi, Phys. Rev. Lett. **92**, 120603 (2004)
  - [9] T. Rizzo cond-mat/0403261.
  - [10] M. Mézard and G. Parisi, Eur. Phys. J. B **20**, 217 (2001)
  - [11] M. Mézard and G. Parisi, J. Stat. Phys **111** (2003) 1
  - [12] M. Mézard, G. Parisi and R. Zecchina, *Science* **297**:812, 2002
  - [13] R. Monasson, Phys. Rev. Lett. **75** (1995) 2847

- [14] A. Montanari and F. Ricci-Tersenghi, Eur. Phys. J. B 33, 339 (2003). A. Montanari, G. Parisi and F. Ricci-Tersenghi, J. Phys. A 37, 2073 (2004).
- [15] A. Cavagna, I. Giardina, G. Parisi, cond-mat/0407440
- [16] T. Rizzo cond-mat/0404729